Govt. Polytechnic Gariyaband
Subject:- Appl. Maths-II
Lecture Notes-2

Numerical Analysis

Example:-1. Find the root of $f(x) = x^3 + x - 1 = 0$ lying in the interval $(0, 1)$ using bisection method up to four iterations. Also find the maximum possible error in the root computed.
Solution:- We have, $f(x) = x^3 + x - 1$.
Since $f(0) = -1$ is negative and $f(1) = 1$ is positive, thus a root of $f(x) = 0$ lies between 0 and 1.
The 1st approximation to the root is $x_1 = \frac{0+1}{2} = 0.5$
We have, $f(0.5) = (0.5)^3 + 0.5 - 1 = -0.375 < 0$, thus the root lies between 0.5 and 1.
The 2nd approximation to the root is $x_2 = \frac{0.5+1}{2} = 0.75$
We have, $f(0.75) = (0.75)^3 + 0.75 - 1 = 0.1719 > 0$, thus the root lies between 0.5 and 0.75.
The 3rd approximation to the root is $x_3 = \frac{0.5+0.75}{2} = 0.625$.
We have, $f(0.625) = (0.625)^3 + 0.625 - 1 = -0.1309 < 0$, thus the root lies between 0.625 and 0.75.
The 4th approximation to the root is $x_4 = \frac{0.625+0.75}{2} = 0.6875$
We have, $f(0.6875) = (0.6875)^3 + 0.6875 - 1 =$
Example:-2. Find an approximate root of the equation $\sin x = \frac{1}{x}$, that lies between $x = 1$and $x = 1.5$ radians using bisection method up to seven iterations.
Solution: Let $f(x) = x \sin x - 1$.
Since $f(1) = 1 \times \sin(1) - 1 = -0.15849$
and, $f(1.5) = 1.5 \times \sin(1.5) - 1 = 0.49625$;
a root lies between 1 and 1.5.
The first approximation to the root is $x_1 = \frac{1+1.5}{2} = 1.25$.
Since $f(x_1) = (1.25)\sin(1.25) - 1 = 0.18627$ and $f(1) < 0$, thus, a root lies between 1 and $x_1 = 1.25$.
The second approximation to the root is $x_2 = \frac{1+1.25}{2} = 1.125$.
Since $f(x_2) = (1.125)\sin(1.125) - 1 = 0.01509$ and $f(1) < 0$, thus, a root lies between 1 and $x_1 = 1.125$.
The third approximation to the root is $x_3 = \frac{1+1.125}{2} = 1.0625$.
Since $f(x_3) = (1.0625)\sin(1.0625) - 1 = -0.0718$ and $f(x_2) > 0$, thus, a root lies between $x_3 = 1.0625$ and $x_2 = 1.125$.
The fourth approximation to the root is $x_4 = \frac{1.0625+1.125}{2} = 1.09375$.
Since $f(x_4) = -0.02836 < 0$ and $f(x_2) > 0$, thus, the root lies between $x_4 = 1.09375$ and $x_2 = 1.125$.
The firth approximation to the root is $x_5 = \frac{1.09375+1.125}{2} = 1.109375$.
Since $f(x_5) = -0.00664 < 0$ and $f(x_2) > 0$, thus, the root lies between $x_5 = 1.10937$ and $x_2 = 1.125$.
The sixth approximation to the root is $x_6 = \frac{1.10937+1.125}{2} = 1.11719$.
Since $f(x_6) = 0.00421 > 0$. But $f(x_5) < 0$, thus, the root lies between $x_5 = 1.10937$ and $x_6 = 1.11719$.
The seventh approximation to the root is $x_7 = \frac{1.10937+1.11719}{2} = 1.11328$.
Hence, the root to the desired approximation is 1.11328.
Example:-3. Find a root of the equation $x^2 - \log_e x - 12 = 0$ upto decimal places by using the bisection method.
Sol.- Let $f(x) = x^2 - \log_e x - 12$

$f(3) = (3)^2 - \log_e 3 - 12 = -4.09861$ (-ve)

$$f(4) = (4)^2 - \log_e 4 - 12 = 2.61371 \quad (+\text{ve})$$

Since value of a function is negative at $x = 3$ and positive at $x = 4$. Therefore root lies between 3 and 4.

First Approximation

Let $a = 3$, $b = 4$

$$x_1 = \frac{3+4}{2} = 3.5$$

$$f(x_1) = f(3.5) = (3.5)^2 - \log_e 3.5 - 12$$
$$= -1.00276 \text{ (-ve)}$$

Root lies between 3.5 and 4.

Second Approximation

$$x_2 = \frac{3.5+4}{2} = 3.75$$

$$f(x_2) = f(3.75) = (3.75)^2 - \log_e 3.75 - 12$$
$$= 0.74074 \text{ (+ve)}$$

Root lies between 3.5 and 3.75.

Third Approximation

$$x_3 = \frac{3.5+3.75}{2} = 3.625$$

$$f(x_3) = f(3.625) = (3.625)^2 - \log_e 3.625 - 12$$
$$= -0.14723 \text{ (-ve)}$$

Root lies between 3.625 and 3.75.

Fourth Approximation

$$x_4 = \frac{3.625+3.75}{2} = 3.6875$$

$$f(x_4) = f(3.6875) = (3.6875)^2 - \log_e 3.6875 - 12$$
$$= 0.29271 \text{ (+ve)}$$

Root lies between 3.625 and 3.6875.

Fifth Approximation

$$x_5 = \frac{3.625+3.6875}{2} = 3.65625$$

$$f(x_5) = f(3.65625) = (3.65625)^2 - \log_e 3.65625 - 12$$
$$= 0.07173 \text{ (+ve)}$$

Root lies between 3.625 and 3.65625.

Sixth Approximation

$$x_6 = \frac{3.625+3.65625}{2} = 3.640625$$

$$f(x_1) = f(3.640625) = (3.640625)^2 - \log_e 3.640625 - 12$$
$$= -0.03800 \text{ (-ve)}$$

Root lies between 3.640625 and 3.65625.

Seventh Approximation

$$x_7 = \frac{3.640625+3.65625}{2} = 3.6484375$$

$$f(x_7) = f(3.6484375) = (3.6484375)^2 - \log_e 3.6484375 - 12$$
$$= 0.0168 \text{ (+ve)}$$

Root lies between 3.640625 and 3.6484375.

Eight Approximation

$$x_8 = \frac{3.640625+3.6484375}{2} = 3.6445312$$

$$f(x_8) = f(3.6445312) = (3.6445312)^2 - \log_e 3.6445312 - 12$$
$$= -0.01062 \text{ (-ve)}$$

Root lies between 3.6445312 and 3.6484375.

Ninth Approximation

$$x_9 = \frac{3.6445312+3.6484375}{2} = 3.6464843$$

$$f(x_9) = f(3.6464843) = (3.6464843)^2 - \log_e 3.6464843 - 12$$
$$= 0.00308 \text{ (-ve)}$$

Root lies between 3.6464843 and 3.6484375.

Tenth Approximation

$$x_{10} = \frac{3.6464843 + 3.6484375}{2} = 3.6474609$$

$$f(x_{10}) = f(3.6474609) = (3.6474609)^2 - \log_e 3.6474609 - 12$$
$$= 0.009939 \text{ (+ve)}$$

Root lies between 3.6464843 and 3.6474609.

Eleventh Approximation

$$x_{11} = \frac{3.6464843 + 3.6474609}{2} = 3.6469726$$

$$f(x_{10}) = f(3.6469726) = (3.6469726)^2 - \log_e 3.6469726 - 12$$
$$= 0.006512 \text{ (+ve)}$$

Root lies between 3.6464843 and 3.6469726.

Hence, the root of the given equation is 3.646 correct to three decimal places after computing eleventh iterations.

Example:-4. Using bisection method find a root of the equation $3x + \sin x = e^x$ correct to three places of decimal.

Solution:- Let $f(x) = 3x + \sin x - e^x$

$$f(0) = 3(0) + \sin 0 - e^0 = -1 \text{ (-ve)}$$
$$f(1) = 3(1) + \sin 1 - e^1 = 1.123189 \quad \text{(+ve)}$$

Since value of a function is negative at $x = 0$ and positive at $x = 1$. Therefore root lies between 0 and 1.

First Approximation

Let $a = 0, b = 1$

$$x_1 = \frac{0 + 1}{2} = 0.5$$

$$f(x_1) = f(0.5) = 3(0.5) + \sin 0.5 - e^{0.5}$$
$$= 0.33070 \text{ (+ve)}$$

Root lies between 0 and 0.5.

Second Approximation

$$x_2 = \frac{0 + 0.5}{2} = 0.25$$

$$f(x_2) = f(0.25) = 3(0.25) + \sin 0.25 - e^{0.25}$$
$$= -0.2866 \text{ (-ve)}$$

Root lies between 0.25 and 0.5.

Third Approximation

$$x_3 = \frac{0.25 + 0.5}{2} = 0.375$$

$$f(x_3) = f(0.375) = 3(0.375) + \sin 0.375 - e^{0.375}$$
$$= 0.03628 \text{ (+ve)}$$

Root lies between 0.25 and 0.375.

Fourth Approximation

$$x_4 = \frac{0.25 + 0.375}{2} = 0.3125$$

$$f(x_4) = f(0.3125) = 3(0.3125) + \sin 0.3125 - e^{0.3125}$$
$$= -0.12189 \text{ (-ve)}$$

Root lies between 0.3125 and 0.375.

Fifth Approximation

$$x_5 = \frac{0.3125 + 0.375}{2} = 0.34375$$

$$f(x_5) = f(0.34375) = 3(0.34375) + \sin 0.34375 - e^{0.34375}$$
$$= -0.04195 \text{ (-ve)}$$

Root lies between 0.34375 and 0.375.

Sixth Approximation

$$x_6 = \frac{0.34375 + 0.375}{2} = 0.359375$$

$$f(x_1) = f(0.359375) = 3(0.359375) + \sin 0.359375 - e^{0.359375}$$
$$= -0.00262 \text{ (-ve)}$$

Root lies between 0.359375 and 0.375.

Seventh Approximation

$$x_7 = \frac{0.359375 + 0.375}{2} = 0.3671875$$
$$f(x_7) = f(0.3671875) = 3(0.3671875) + \sin 0.3671875 - e^{0.3671875}$$
$$= 0.0168 \text{ (+ve)}$$

Root lies between 0.359375 and 0.3671875.

Eight Approximation

$$x_8 = \frac{0.359375 + 0.3671875}{2} = 0.36328$$
$$f(x_8) = f(0.36328) = 3(0.36328) + \sin 0.36328 - e^{0.36328}$$
$$= 0.007144 \text{ (+ve)}$$

Root lies between 0.359375 and 0.36328.

Ninth Approximation

$$x_9 = \frac{0.359375 + 0.36328}{2} = 0.3613275$$
$$f(x_9) = f(0.3613275) = (0.3613275)^2 - \log_e 0.3613275 - 12$$
$$= 0.00226 \text{ (+ve)}$$

Root lies between 0.359375 and 0.3613275.

Tenth Approximation

$$x_{10} = \frac{0.359375 + 0.3613275}{2} = 0.36035125$$
$$f(x_{10}) = f(0.36035125) = (0.36035125)^2 - \log_e 0.36035125 - 12$$
$$= -0.000176 \text{ (-ve)}$$

Root lies between 0.359375 and 0.36035125.

Eleventh Approximation

$$x_{11} = \frac{0.359375 + 0.36035125}{2} = 0.359863$$
$$f(x_{11}) = f(0.359863) = (0.359863)^2 - \log_e 0.359863 - 12$$
$$= -0.00139 \text{ (-ve)}$$

Root lies between 0.359863 and 0.36035125.

Twelfth Approximation

$$x_{12} = \frac{0.359863 + 0.36035125}{2} = 0.360107$$
$$f(x_{12}) = f(0.360107) = (0.360107)^2 - \log_e 0.360107 - 12$$
$$= -0.000787 \text{ (-ve)}$$

Root lies between 0.360107 and 0.36035125.

Hence, the root of the given equation is 0.360 correct to three decimal places after computing twelfth iterations.

Example:-5. Apply bisection method to determine the approximate root of the equation $x \log_{10} x - 1.2 = 0$ which lies between 2 and 3.

Solution:- Let $f(x) = x \log_{10} x - 1.2$

$$f(2) = (2)\log_{10} 2 - 1.2 = -0.59794 \text{ (-ve)}$$
$$f(3) = (3)\log_{10} 3 - 1.2 = 0.23136 \text{ (+ve)}$$

Since value of a function is negative at $x = 2$ and positive at $x = 3$. Therefore root lies between 2 and 3.

First Approximation

Let $a = 2$, $b = 3$

$$x_1 = \frac{2 + 3}{2} = 2.5$$
$$f(x_1) = f(2.5) = (2.5)\log_{10} 2.5 - 1.2$$
$$= -0.20515 \text{ (-ve)}$$

Root lies between 2.5 and 3.

Second Approximation

$$x_2 = \frac{2.5 + 3}{2} = 2.75$$
$$f(x_2) = f(2.75) = (2.75)\log_{10} 2.75 - 1.2$$
$$= 0.00816 \text{ (+ve)}$$

Root lies between 2.5 and 2.75.

Third Approximation

$$x_3 = \frac{2.5 + 2.75}{2} = 2.625$$
$$f(x_3) = f(2.625) = (2.625)\log_{10} 2.625 - 1.2$$
$$= -0.09978 \text{ (-ve)}$$

Root lies between 2.625 and 2.75.

Fourth Approximation

$$x_4 = \frac{2.625 + 2.75}{2} = 2.6875$$
$$f(x_4) = f(2.6875) = (2.6875)\log_{10} 2.6875 - 1.2$$
$$= -0.046126 \text{ (-ve)}$$

Root lies between 2.6875 and 2.75.

Fifth Approximation

$$x_5 = \frac{2.6875 + 2.75}{2} = 2.71875$$
$$f(x_5) = f(2.71875) = (2.71875)\log_{10} 2.71875 - 1.2$$
$$= -0.019058 \text{ (-ve)}$$

Root lies between 2.71875 and 2.75.

Sixth Approximation

$$x_6 = \frac{2.71875 + 2.75}{2} = 2.734375$$
$$f(x_6) = f(2.734375) = (2.734375)\log_{10} 2.734375 - 1.2$$
$$= -0.0054662 \text{ (-ve)}$$

Root lies between 2.734375 and 2.75.

Seventh Approximation

$$x_7 = \frac{2.734375 + 2.75}{2} = 2.7421875$$
$$f(x_7) = f(2.7421875) = (2.7421875)\log_{10} 2.7421875 - 1.2$$
$$= 0.001344 \text{ (+ve)}$$

Root lies between 2.734375 and 2.7421875.

Eighth Approximation

$$x_8 = \frac{2.734375 + 2.7421875}{2} = 2.73828125$$
$$f(x_8) = f(2.73828125) = (2.73828125)\log_{10} 2.73828125 - 1.2$$
$$= -0.002062 \text{ (-ve)}$$

Root lies between 2.73828125 and 2.7421875.

Ninth Approximation

$$x_9 = \frac{2.73828125 + 2.7421875}{2} = 2.740234375$$
$$f(x_9) = f(2.740234375) = (2.740234375)\log_{10} 2.740234375 - 1.2$$
$$= -0.000359 \text{ (-ve)}$$

Root lies between 2.740234375 and 2.7421875.

Hence, the root of the given equation is 2.74 correct to two decimal places after computing ninth iterations.

Example:-6. Using bisection method, find the cube root of 25.

Solution:- Let $x = (25)^{1/3}$

$$x^3 = 25$$
$$x^3 - 25 = 0$$
$$f(x) = x^3 - 25$$
$$f(2) = 2^3 - 25 = -17 \quad \text{(-ve)}$$
$$f(3) = 3^3 - 25 = 2 \quad \text{(+ve)}$$

Root lies between 2 and 3.

First Approximation
Let $a = 2, b = 3$

$$x_1 = \frac{2+3}{2} = 2.5$$
$$f(x_1) = f(2.5) = 2^3 - 25$$
$$= -9.375 \text{ (-ve)}$$

Root lies between 2.5 and 3.

Second Approximation

$$x_2 = \frac{2.5+3}{2} = 2.75$$
$$f(x_2) = f(2.75) = (2.75)^3 - 25$$
$$= -4.203125 \text{ (+ve)}$$

Root lies between 2.75 and 3.

Third Approximation

$$x_3 = \frac{2.75+3}{2} = 2.875$$
$$f(x_3) = f(2.875) = (2.875)^3 - 25$$
$$= -1.236 \text{ (-ve)}$$

Root lies between 2.875 and 3.

Fourth Approximation

$$x_4 = \frac{2.875+3}{2} = 2.9375$$
$$f(x_4) = f(2.9375) = (2.9375)^3 - 25$$
$$= 0.34 \text{ (+ve)}$$

Root lies between 2.875 and 2.9375.

Fifth Approximation

$$x_5 = \frac{2.9375+2.875}{2} = 2.90625$$
$$f(x_5) = f(2.90625) = (2.90625)^3 - 25$$
$$= -0.45 \text{ (-ve)}$$

Root lies between 2.90625 and 2.9375.

Sixth Approximation

$$x_6 = \frac{2.90625+2.9375}{2} = 2.921875$$
$$f(x_1) = f(2.921875) = (2.921875)^3 - 25$$
$$= -0.054 \text{ (-ve)}$$

Root lies between 2.921875 and 2.9375.

Seventh Approximation

$$x_7 = \frac{2.921875+2.9375}{2} = 2.9296875$$
$$f(x_7) = f(2.9296875) = (2.9296875)^3 - 25$$
$$= 0. \text{ (+ve)}$$

Root lies between 2.921875 and 2.9296875.

Hence, the root of the given equation is 2.92 correct to two decimal places after computing seventh iterations.

Example 7. Compute the root of the equation $x^3 - x - 11 = 0$ by bisection method.

Solution- Let $f(x) = x^3 - x - 11$

$$f(2) = (2)^3 - 2 - 11 = -5 \quad \text{(-ve)}$$
$$f(3) = (3)^3 - 3 - 11 = 13 \quad \text{(+ve)}$$

The root lies between 2 and 3.

First Approximation
Let $a = 2, b = 3$

$$x_1 = \frac{2+3}{2} = 2.5$$
$$f(x_1) = f(2.5) = (2.5)^3 - 2.5 - 11$$
$$= 2.125 \text{ (+ve)}$$

Root lies between 2 and 2.5.

Second Approximation

$$x_2 = \frac{2 + 2.5}{2} = 2.25$$
$$f(x_2) = f(2.25) = (2.25)^3 - 2.25 - 11$$
$$= -1.859 \text{ (+ve)}$$

Root lies between 2.25 and 2.5.

Third Approximation

$$x_3 = \frac{2.5 + 2.25}{2} = 2.375$$
$$f(x_3) = f(2.375) = (2.375)^3 - 2.375 - 11$$
$$= 0.021 \text{ (+ve)}$$

Root lies between 2.25 and 2.375.

Fourth Approximation

$$x_4 = \frac{2.25 + 2.375}{2} = 2.3125$$
$$f(x_4) = f(2.3125) = (2.3125)^3 - 2.3125 - 11$$
$$= -0.94 \text{ (-ve)}$$

Root lies between 2.3125 and 2.375.

Fifth Approximation

$$x_5 = \frac{2.3125 + 2.375}{2} = 2.34375$$
$$f(x_5) = f(2.34375) = (2.34375)^3 - 2.34375 - 11$$
$$= -0.46 \text{ (-ve)}$$

Root lies between 2.34375 and 2.375.

Sixth Approximation

$$x_6 = \frac{2.34375 + 2.375}{2} = 2.359375$$
$$f(x_6) = f(2.359375) = (2.359375)^3 - 2.359375 - 11$$
$$= -0.22 \text{ (-ve)}$$

Root lies between 2.359375 and 2.375.

Seventh Approximation

$$x_7 = \frac{2.359375 + 2.375}{2} = 2.3671875$$
$$f(x_7) = f(2.3671875) = (2.3671875)^3 - 2.3671875 - 11$$
$$= 0.1. \text{ (+ve)}$$

Root lies between 2.359375 and 2.3671875.

Example:-1. Apply false position method to find a root of $f(x) = x^2 - x - 2 = 0$ in the interval (1, 3) up to three iterations.

Solution:- Here $x_0 = 1,\ x_1 = 3$ and $f(x) = x^2 - x - 2$.

We have, $f(1) = -2$ and $f(3) = 4$, thus, a root of $f(x) = 0$ lies between 1 and 3.

The first approximation is

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$
$$= 1 - \frac{3-1}{4-(-2)}(-2) = 1.667.$$

We have, $f(1.667) = -0.8889 < 0$, thus, the root lies between 1.667 and 3.

The second approximation is

$$x_3 = 1.667 - \frac{3-1.667}{4-(-0.8889)}(-0.8889) = 1.909.$$

We have, $f(1.909) = -0.2647 < 0$, thus, the root lies between 1.909 and 3.

The third approximation is

$$x_4 = 1.909 - \frac{3-1.909}{4-(-0.2647)}(-0.2647)$$

$$= 1.909 + \frac{1.091}{4.2647}(-0.2647)$$

$$= 1.9767.$$

Example:-2. Use the method of false position, to find the fourth root of 32 correct to three decimal places.

Solution:- Let $x = (32)^{1/4}$, so that $x^4 - 32 = 0$.

Take $f(x) = x^4 - 32$. Then $f(2) = -16$ and $f(3) = 49$, thus, a root lies beteen 2 and 3.

Taking $x_0 = 2, x_1 = 3, f(x_0) = -16, f(x_1) = 49$ in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2 + \frac{16}{15} = 2.2462$$

Now, $f(x_2) = f(2.2462) = -6.5438$, thus, the root lies between 2.2462 and 3.

Next taking $x_0 = 2.2462, x_1 = 3, f(x_0) = -6.5438, f(x_1) = 49$, we get

$$x_3 = 2.2462 - \frac{3 - 2.2462}{49 + 6.5438}(-6.5438) = 2.335$$

Now, $f(x_3) = f(2.335) = -2.2732$, thus, the root lies between 2.335 and 3.

Next taking $x_0 = 2.335$ and $x_1 = 3, f(x_0) = -2.2732$ and $f(x_1) = 49$, we obtain

$x_4 = 2.335 - \frac{3-2.335}{49+2.2732}(-2.2732) = 2.3645$.

Repeating this process, the successive approximations are $x_5 = 2.3770, x_6 = 2.3779$ and since $x_5 = x_6$ upto 3 decimal places, we take $(32)^{1/4} = 2.378$.

Example:-3. Find the root of the equation $x^3 - x^2 - 2 = 0$ by False position method.

Solution:- we have, $f(x) = x^3 - x^2 - 2$

$$f(1) = (1)^3 - (1)^2 - 2 = -2$$

$$f(2) = (2)^3 - (2)^2 - 2 = 2$$

Taking $x_0 = 1$ and $x_1 = 2$ then $f(x_0) = -2$ and $f(x_1) = 2$.

By False Position method the first approximation is

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1 - \frac{2-1}{2-(-2)}(-2)$$

$$= 1 + 0.5$$

$$= 1.5.$$

$$f(x_2) = f(1.5) = (1.5)^3 - (1.5)^2 - 2$$

$$= -0.875 \quad \text{(-ve)}$$

Root lies between 1.5 and 2.

Second approximation

Taking $x_0 = 1.5$ and $x_1 = 2$ then $f(x_0) = -0.875$ and $f(x_1) = 2$

$$x_3 = 1.5 - \frac{2-1.5}{2-(-0.875)}(-0.875)$$

$$= 1.5 + \frac{0.4375}{2.875}$$

$$= 1.5 + 0.152173$$

$$= 1.652173$$

$$f(x_3) = f(1.652173) = (1.652173)^3 - (1.652173)^2 - 2$$

$$= -0.21977926 \quad \text{(-ve)}$$

Root lies between 1.652173 and 2.

Third approximation

Taking $x_0 = 1.652173$ and $x_1 = 2$ then $f(x_0) = -0.21977926$ and $f(x_1) = 2$

$$x_4 = 1.652173 - \frac{2-1.652173}{2-(-0.21977926)}(-0.21977926)$$

$$= 1.652173 + \frac{0.07644516}{2.21977926}$$

$$= 1.652173 + 0.03443818$$

$$= 1.68661118$$

$$f(x_4) = f(1.68661118) = (1.68661118)^3 - (1.68661118)^2 - 2$$

$$= -0.0468265 \quad \text{(-ve)}$$

Root lies between 1.68661118 and 2.

Fourth approximation

Taking $x_0 = 1.68661118$ and $x_1 = 2$ then $f(x_0) = -0.0468265$ and $f(x_1) = 2$.

$$x_5 = 1.68661118 - \frac{2-1.68661118}{2-(-0.0468265)}(-0.0468265)$$

$= 1.68661118 + \frac{0.0146749}{2.0468265}$

$= 1.68661118 + 0.00716958$

$= 1.6937807$

$f(x_5) = f(1.6937807) = (1.6937807)^3 - (1.6937807)^2 - 2$

$= -0.0096173$ (-ve)

Root lies between 1.6937807 and 2.

Fourth approximation

Taking $x_0 = 1.6937807$ and $x_1 = 2$ then $f(x_0) = -0.0096173$ and $f(x_1) = 2$.

$x_6 = 1.6937807 - \frac{2-1.6937807}{2-(-0.0096173)}(-0.0096173)$

$= 1.6937807 + \frac{0.002945}{2.0096173}$

$= 1.6937807 + 0.0014653$

$= 1.695246$

The value of x_5 and x_6 are same up to two places of decimal. Hence root is 1.69.

Example:-4. Using False position method, find the root of the equation $x^3 - 9x + 1 = 0$ which lies between 2 and 4.

Solution:- we have, $f(x) = x^3 - 9x + 1$

$f(2) = (2)^3 - 9(2) + 1 = -9$ (-ve)

$f(4) = (4)^3 - 9(4) + 1 = 29$ (+ve)

Taking $x_0 = 2$ and $x_1 = 4$ then $f(x_0) = -9$ and $f(x_1) = 29$.

By False Position method the first approximation is

$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$

$= 2 - \frac{4-2}{29-(-9)}(-9)$

$=$

$= 2.47368.$

$f(x_2) = f(2.47368) = (2.47368)^3 - 9(2.47368) + 1$

$= -6.1264426$ (-ve)

Root lies between 2.47368 and 4.

Second approximation

Taking $x_0 = 2.47368$ and $x_1 = 4$ then $f(x_0) = -6.1264426$ and $f(x_1) = 29$

$x_3 = 2.47368 - \frac{4-2.47368}{29-(-6.1264426)}(-6.1264426)$

$= 2.47368 + -$

$= 2.47368 +$

$= 2.7398871$

$f(x_3) = f(2.7398871) = (2.7398871)^3 - 9(2.7398871) + 1$

$= -3.090702$ (-ve)

Root lies between 2.7398871 and 4.

Third approximation

Taking $x_0 = 2.7398871$and $x_1 = 4$ then $f(x_0) = -3.090702$ and $f(x_1) = 29$

$x_4 = 2.7398871 - \frac{4-2.7398871}{29-(-3.090702)}(-3.090702)$

$= 2.7398871 + -$

$= 2.7398871 +$

$= 2.861250$

$f(x_4) = f(2.861250) = (2.861250)^3 - 9(2.861250) + 1$

$= -1.326907$ (-ve)

Root lies between 2.861250 and 4.

Fourth approximation

Taking $x_0 = 2.861250$ and $x_1 = 4$ then $f(x_0) = -1.326907$ and $f(x_1) = 29$.

$x_5 = 2.861250 - \frac{4-2.861250}{29-(-1.326907)}(-1.326907)$

$= 2.861250 + -$

$= 2.861250 +$

$= 2.911074$

$f(x_5) = f(2.911074) = (2.911074)^3 - 9(2.911074) + 1$

$= -0.530200$ (-ve)

Root lies between 2.911074 and 4.

Fifth approximation

Taking $x_0 = 2.911074$ and $x_1 = 4$ then $f(x_0) = -0.530200$ and $f(x_1) = 29$.

$$x_6 = 2.911074 - \frac{4-2.911074}{29-(-0.530200)}(-0.530200)$$

$= 2.911074 + -$

$= 2.911074 +$

$= 2.9306251$

$f(x_6) = f(2.9306251) = (2.9306251)^3 - 9(2.9306251) + 1$

$= -0.2057662$ (-ve)

Root lies between 2.911074 and 4.

Sixth approximation

Taking $x_0 = 2.9306251$ and $x_1 = 4$ then $f(x_0) = -0.2057662$ and $f(x_1) = 29$.

$$x_7 = 2.9306251 - \frac{4-2.9306251}{29\ \ (-0.2057662)}(-0.2057662)$$

$= 2.9306251 + -$

$= 2.9306251 +$

$= 2.9381592$

$f(x_7) = f(2.9381592) = (2.9381592)^3 - 9(2.9381592) + 1$

$= -0.078952$ (-ve)

Root lies between 2.9381592 and 4.

Seventh approximation

Taking $x_0 = 2.9381592$ and $x_1 = 4$ then $f(x_0) = -0.078952$ and $f(x_1) = 29$.

$$x_8 = 2.9381592 - \frac{4-2.9381592}{29-(-0.078952)}(-0.078952)$$

$= 2.9381592 + -$

$= 2.9381592 +$

$= 2.94092$

$f(x_8) = f(2.9381592) = (2.9381592)^3 - 9(2.9381592) + 1$

$= -0.078952$ (-ve)

Root lies between 2.9381592 and 4.

Eighth approximation

Taking $x_0 = 2.9381592$ and $x_1 = 4$ then $f(x_0) = -0.078952$ and $f(x_1) = 29$.

$$x_9 = 2.9381592 - \frac{4-2.9381592}{29-(-0.078952)}(-0.078952)$$

$= 2.9381592 + -$

$= 2.9381592 +$

$= 2.94092$

$f(x_9) = f(2.94092) = (2.94092)^3 - 9(2.94092) + 1$

$= -0.032232$ (-ve)

Root lies between 2.94092 and 4.

Ninth approximation

Taking $x_0 = 2.94092$ and $x_1 = 4$ then $f(x_0) = -0.032232$ and $f(x_1) = 29$.

$$x_{10} = 2.94092 - \frac{4-2.94092}{29\ \ (-0.032232)}(-0.032232)$$

$= 2.94092 + -$

$= 2.940922 +$

$= 2.942095$

$f(x_{10}) = f(2.942095) = (2.942095)^3 - 9(2.942095) + 1$

$= -0.012307$ (-ve)

Root lies between 2.9381592 and 4.

Tenth approximation

Taking $x_0 = 2.942095$ and $x_1 = 4$ then $f(x_0) = -0.012307$ and $f(x_1) = 29$.

$$x_{11} = 2.942095 - \frac{4-2.942095}{29-(-0.012307)}(-0.012307)$$
$$= 2.942095 + -$$
$$= 2.942095 +$$
$$= 2.942543$$
The value of x_{10} and x_{11} are same upto three places of decimal. Hence root is 2.942.

Example:-1. By applying Newton's method upto two iterations, find the real root near to 2 for the equation $x^4 - 12x + 7 = 0$.
Solution:- Let $f(x) = x^4 - 12x + 7$
This gives, $f'(x) = 4x^3 - 12$
Here $x_0 = 2$. Therefore,
$f(x_0) = f(2) = 2^4 - 12 \times 2 + 7 = -1$, and
$f'(x_0) = f'(2) = 4(2)^3 - 12 = 20$
Using Newton-Raphson formula, we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 2 - \frac{(-1)}{20} = \frac{41}{20} = 2.05$$

And further

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 2.05 - \frac{(2.05)^4 - 1\ (2.05) + 7}{4(2.05)^2 - 12}$$
$$= 2.6706$$

Thus, the root of the equation is 2.6706.
Example-2. Find using Newton's method, the root of the equation $e^x = 4x$, near to 2, correct to three places of decimals.
Solution- We have, $f(x) = e^x - 4x$

$$f(2) = e^2 - 8 = 7.389056 - 8 = -0.610944 \text{ (-ve)}$$
$$f(3) = e^3 - 12 = 20.085537 - 12 = 8.085537 \text{ (+ve)}$$

Since $f(2)(3) < 0$, thus, $f(x) = 0$ has a root between 2 and 3.
Let $x_0 = 2.1$ be the approximate value of the root of the equation given.
Now, $f(x) = e^x - 4x$ gives $f'(x) = e^x - 4$
Therefore, $f(x_0) = e^{2.1} - 4(2.1) = 8.16617 - 8.4 = -0.23383$, and
$f'(x_0) = e^{2.1} - 4 = 4.16617$.
Let x_1 be the first approximation of the root, then by Newton-Raphson formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 2.1 - \frac{(-0.23383)}{4.16617}$$
$$= 2.1 + 0.0561258$$
$$= 2.1561 \text{ (approximately)}$$

Let x_2 denotes the second approximation, then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 2.1561 - \frac{[e^{2.1561} - 4(2.1561)]}{[e^{2.1561} - 4]}$$
$$= 2.1561 - \frac{0.0129861}{4.6373861}$$
$$= 2.1561 - 0.0028003$$
$$= 2.1533 \text{ (approximately)}$$
$$f(x_3) = f(2.1533) = -0.0013484$$
$$f'(x_4) = f'(2.1533) = 4.6106516.$$

Let x_3 denotes the third approximation, then

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 2.1533 - \frac{(-0.013484)}{4.6106516}$$
$$= 2.1532$$

Thus, the value of the root correct to three decimal places is 2.153.

Example:-3. Using Newton-Raphson method, find the root of the equation $e^x - 4x = 0$ correct to three places of decimal.

Solution:-We have, $f(x) = e^x - 4x$

$$f(2) = e^2 - 8 = -0.610944 \text{ (-ve)}$$
$$f(3) = e^3 - 12 = 8.085537 \text{ (+ve)}$$

Since $f(2)f(3) < 0$, therefore root lies between 2 and 3.

Let $x_0 = 2.4$ be the approximate value of the root of the equation.

Now, $f(x) = e^x - 4x$

$f'(x) = e^x - 4$

Let x_1 be the first approximation of the root, then by Newton-Raphson formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 2.4 - \frac{e^{2.4} - 4(2.4)}{e^{2.4} - 4}$$
$$= 2.4 - \frac{1.42318}{7.02318}$$
$$= 2.4 - 0.202640$$
$$= 2.19736$$

Let x_2 denotes the second approximation of the root, then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 2.19736 - \frac{e^{2.19736} - 4(2.19736)}{e^{2.19736} - 4}$$
$$= 2.19736 - \frac{0.211779}{5.001219}$$
$$= 2.19736 - 0.042345$$
$$= 2.155015.$$

Let x_3 denotes the third approximation of the root, then

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 2.155015 - \frac{e^{2.155015} - 4(2.155015)}{e^{2.155015} - 4}$$
$$= 2.155015 - \frac{0.007959}{4.628019}$$
$$= 2.155015 - 0.001719$$
$$= 2.153296.$$

Let x_4 denotes the fourth approximation of the root, then

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= 2.153296 - \frac{e^{2.153296} - 4(2.153296)}{e^{2.153296} - 4}$$
$$= 2.153296 - \frac{0.000016}{4.61320}$$
$$= 2.153296 - 0.000003$$
$$= 2.153293.$$

Thus, the value of the root correct to five decimal places is 2.15329.

Example:-4. Using the method of Newton-Raphson, find the root of the equation $x^3 - 5x + 3 = 0$.

Solution:- We have, $f(x) = x^3 - 5x + 3$

$$f(0) = 0 - 0 + 3 = 3 \qquad \text{(+ve)}$$
$$f(1) = 1^3 - 5(1) + 3 = -1 \quad \text{(-ve)}$$

Since $f(0)f(1) < 0$ therefore root lies between 0 and 1.

Let $x_0 = 0.7$ be the approximate value of the root of the equation.

Now, $f(x) = x^3 - 5x + 3$

$f'(x) = 3x^2 - 5$

Then by Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^3-5x_n+3}{3x_n^2-5}$$

$$x_{n+1} = \frac{3x_n^3-5x_n-x_n^3+5x_n-3}{3x_n^2-5}$$

$$x_{n+1} = \frac{2x_n^3-3}{3x_n^2-5} \qquad ....(1)$$

For first approximation, putting $n = 0$ in equation (1)

$$x_1 = \frac{2x_0^3-3}{3x_0^2-5}$$

$$= \frac{2(0.7)^3-3}{3(0.7)^2-5}$$

$$x_1 = 0.655524$$

For second approximation, putting $n = 1$ in equation (1)

$$x_2 = \frac{2x_1^3-3}{3x_1^2-5}$$

$$= \frac{2(0.655524)^3-3}{3(0.655524)^2-5}$$

$$= 0.6566197$$

For third approximation, putting $n = 2$ in equation (1)

$$x_3 = \frac{2x_2^3-3}{3x_2^2-5}$$

$$= \frac{2(0.6566197)^3-3}{3(0.6566197)^2-5}$$

$$= 0.656620$$

For fourth approximation, putting $n = 3$ in equation (1)

$$x_4 = \frac{2x_3^3-3}{3x_3^2-5}$$

$$= \frac{2(0.656620)^3-3}{3(0.656620)^2-5}$$

$$= 0.6566204$$

The value of x_3 and x_4 are same upto six places of decimal. Hence root is 0.656620.

Example:-5. Using the method of Newton-Raphson, find the root of the equation $2x - 5 - 3\sin x = 0$.

Solution:- We have, $f(x) = 2x - 5 - 3\sin x$

$$f(2) = 2(2) - 5 - 3\sin 2 = -3.72789 \qquad \text{(-ve)}$$

$$f(3) = 2(3) - 5 - 3\sin 3 = 0.57664 \qquad \text{(+ve)}$$

Since $f(2)f(3) < 0$ therefore root lies between 2 and 3.

Let $x_0 = 2.5$ be the approximate value of the root of the equation.

Now, $f(x) = 2x - 5 - 3\sin x$

$f'(x) = 2 - 3\cos x$

Then by Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{2x_n-5-3\sin x_n}{2-3\cos x_n}$$

$$x_{n+1} = \frac{2x_n-3x_n\cos x_n-2x_n+5+3\sin x_n}{2-3\cos x_n}$$

$$x_{n+1} = \frac{5+3\sin x_n-3x_n\cos x_n}{2-3\cos x_n} \qquad ....(1)$$

For first approximation, putting $n = 0$ in equation (1)

$$x_1 = \frac{5+3\sin x_0-3x_0\cos x_0}{2-3\cos x_0}$$

$$= \frac{5+3\sin(2.5)-3(2.5)\cos(2.5)}{2-3\cos(2.5)}$$

$$x_1 = 2.907731266$$

For second approximation, putting $n = 1$ in equation (1)

$$x_2 = \frac{5+3\sin x_1 - 3x_0\cos x_1}{2-3\cos x_1}$$

$$= \frac{5+3\,\text{si}\ \ (2.907731266)-3(2.907731266)\cos(2.907731266)}{2-3\,\text{co}\ \ (2.907731266)}$$

$= 2.883280727$

For third approximation, putting $n = 2$ in equation (1)

$$x_3 = \frac{5+3\sin x_2 - 3x_0\cos x_2}{2-3\cos x_2}$$

$$= \frac{5+3\,\text{si}\ \ (2.883280727)-3(2.883280727)\cos(2.883280727)}{2-3\cos(2.883280727)}$$

$= 2.883236873$

For fourth approximation, putting $n = 3$ in equation (1)

$$x_4 = \frac{5+3\sin x_3 - 3x_0\cos x_3}{2-3\cos x_3}$$

$$= \frac{5+3\sin(2.883236873)-3(2.883236873)\cos(2.883236873)}{2-3\,\text{co}\ \ (2.883236873)}$$

$= 2.883236873.$

The value of x_3 and x_4 are same up to nine places of decimal. Hence root is 2.883236873.

Example:-6. Using the method of Newton-Raphson, find the root of the equation $\cos x - xe^x = 0$.

Solution:- We have, $f(x) = \cos x - xe^x$

$f(0) = \cos(0) - (0)e^0 = 1$ (+ve)

$f(1) = \cos(1) - (1)e^1 = -2.177979$ (-ve)

Since $f(0)f(1) < 0$ therefore root lies between 0 and 1.

Let $x_0 = 0.4$ be the approximate value of the root of the equation.

Now, $f(x) = \cos x - xe^x$

$f'(x) = -\sin x - xe^x - e^x$

Then by Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{\cos x_n - x_n e^{x_n}}{-\sin x_n - x_n e^{x_n} - e^{x_n}}$$

$$x_{n+1} = \frac{-x_n\sin x_n - x_n^2 e^{x_n} - x_n e^{x_n} - \cos x_n + x_n e^{x_n}}{-\sin x_n - x_n e^{x_n} - e^{x_n}}$$

$$x_{n+1} = \frac{x_n\sin x_n + x_n^2 e^{x_n} + \cos x_n}{\sin x_n + x_n e^{x_n} + e^{x_n}} \qquad ....(1)$$

For first approximation, putting $n = 0$ in equation (1)

$$x_1 = \frac{x_0\sin x_0 + x_0^2 e^{x_0} + \cos x_0}{\sin x_0 + x_0 e^{x_0} + e^{x_0}}$$

$$= \frac{(0.4)\sin(0.4)+(0.4)^2 e^{(0.4)}+\cos(0.4)}{\sin(0.4)+(0.4)e^{(0.4)}+e^{(0.4)}}$$

$x_1 = 0.530885657$

For second approximation, putting $n = 1$ in equation (1)

$$x_2 = \frac{x_1\sin x_1 + x_1^2 e^{x_1} + \cos x_1}{\sin x_1 + x_1 e^{x_1} + e^{x_1}}$$

$$= \frac{(0.530885657)\sin(0.530885657)+(0.530885657)^2 e^{(0.530885657)}+\cos(0.530885657)}{\sin(0.530885657)+(0.530885657)e^{(0.530885657)}+e^{(0.530885657)}}$$

$= 0.517899869$

For third approximation, putting $n = 2$ in equation (1)

$$x_3 = \frac{x_2\sin x_2 + x_2^2 e^{x_2} + \cos x_2}{\sin x_2 + x_2 e^{x_2} + e^{x_2}}$$

$$= \frac{(0.517899869)\sin(0.517899869)+(0.517899869)^2 e^{(0.517899869)}+\cos(0.517899869)}{\sin(0.517899869)+(0.517899869)e^{(0.517899869)}+e^{(0.517899869)}}$$

$= 0.51775738$

For fourth approximation, putting $n = 3$ in equation (1)

$$x_4 = \frac{x_3\sin x_3 + x_3^2 e^{x_3} + \cos x_3}{\sin x_3 + x_3 e^{x_3} + e^{x_3}}$$

$$= \frac{(0.51775738)\sin(0.51775738)+(0.51775738)^2 e^{(0.51775738)}+\cos(0.51775738)}{\sin(0.51775738)+(0.51775738)e^{(0.51775738)}+e^{(0.51775738)}}$$

$$= 0.517757363.$$

The value of x_3 and x_4 are same up to seven places of decimal. Hence root is 0.5177573.

Example:-7. Find the square root of the 35 by Newton-Raphson method.

Solution:- Let $x = \sqrt{35}$

$$x^2 - 35 = 0$$

Let $f(x) = x^2 - 35$

$f(5) = 5^2 - 35 = -10$ (-ve)

$f(6) = 6^2 - 35 = 1$ (+ve)

Since $f(5)f(6) < 0$ therefore root lies between 5 and 6.

Let $x_0 = 5.5$ be the approximate value of the root of the equation.

Now, $f(x) = x^2 - 35$

$f'(x) = 2x$

Then by Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 35}{2x_n}$$

$$x_{n+1} = \frac{2x_n^2 - x_n^2 + 3}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + 35}{2x_n} \quad ....(1)$$

For first approximation, putting $n = 0$ in equation (1)

$$x_1 = \frac{x_0^2 + 35}{2x_0}$$

$$= \frac{(5.5)^2 + 35}{2(5.5)}$$

$$x_1 = 5.931818182$$

For second approximation, putting $n = 1$ in equation (1)

$$x_2 = \frac{x_1^2 + 35}{2x_1}$$

$$= \frac{(5.931818182)^2 + 3}{2(5.931818182)}$$

$$x_2 = 5.916100662$$

For third approximation, putting $n = 2$ in equation (1)

$$x_3 = \frac{x_2^2 + 35}{2x_2}$$

$$= \frac{(5.916100662)^2 + 35}{2(5.916100662)}$$

$$x_3 = 5.916079783$$

For fourth approximation, putting $n = 3$ in equation (1)

$$x_4 = \frac{x_3^2 + 35}{2x_3}$$

$$= \frac{(5.916079783)^2 + 35}{2(5.916079783)}$$

$$x_4 = 5.916079783$$

The value of x_3 and x_4 are same up to nine places of decimal. Hence root is 5.916079783.